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The switching element for a Leonard pair

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Abstract

Let V denote a vector space with finite positive dimension. We consider a pair of linear transformations $A: V \to V$ and $A^*: V \to V$ that satisfy (i) and (ii) below:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

We call such a pair a *Leonard pair* on V. Let $\{v_i\}_{i=0}^d$ (resp. $\{w_i\}_{i=0}^d$) denote a basis for V referred to in (i) (resp. (ii)). We show that there exists a unique linear transformation $S: V \to V$ that sends v_0 to a scalar multiple of v_d, fixes w_0, and sends w_i to a scalar multiple of w_i for $1 \leqslant i \leqslant d$. We call S the *switching element*. We describe S from many points of view.

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1. Leonard pairs

We begin by recalling the notion of a Leonard pair. We will use the following terms. A square matrix X is said to be *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume X is tridiagonal. Then X is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. We now define a Leonard pair. For the rest of this paper $\mathbb{K}$ will denote a field.

Definition 1.1 [29]. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair A, A^* where $A: V \to V$ and $A^*: V \to V$ are linear transformations that satisfy (i) and (ii) below:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Note 1.2. It is a common notational convention to use A^* to represent the conjugate-transpose of A. We are *not* using this convention. In a Leonard pair A, A^* the linear transformations A and A^* are arbitrary subject to (i) and (ii) above.

We refer the reader to [5,16,19,20,21,22,23,24,27,28,29,31,32,33,34,35,36,37,38,40,41,42] for background on Leonard pairs. We especially recommend the survey [38]. See [1,2,3,4,6,7,8,9,10, 11,12,13,14,17,18,25,26,30,39,43] for related topics.

We now give an informal summary of the present paper; our formal treatment will begin in Section 2. In [35, Theorem 6.3] and [36, Theorem 3.2] some characterizations of Leonard pairs were given, and in these characterizations a certain invertible matrix was used. Our goal in this paper is to systematically investigate this matrix, or more precisely, the corresponding linear transformation which we denote by S. We define S as follows. Let A, A^* denote the Leonard pair on V from Definition 1.1. Let $\{v_i\}_{i=0}^d$ (resp. $\{w_i\}_{i=0}^d$) denote a basis for V referred to in part (i) (resp. (ii)) of that definition. We show that there exists a unique linear transformation $S: V \to V$ that sends v_0 to a scalar multiple of v_d, fixes w_0, and sends w_i to a scalar multiple of w_i for $1 \leqslant i \leqslant d$. We call S the *switching element* for A, A^*. We show S is invertible. There is a well-known correspondence between Leonard pairs and sequences of orthogonal polynomials from the terminating branch of the Askey scheme [15,38]; we show that $S = u_d(A)$ where $\{u_i\}_{i=0}^d$ are the polynomials that correspond to A, A^*. A *flag* on V is a sequence $\{F_i\}_{i=0}^d$ of subspaces of V such that F_i has dimension $i+1$ for $0 \leqslant i \leqslant d$ and $F_{i-1} \subseteq F_i$ for $1 \leqslant i \leqslant d$. Following [31, Definition 7.2] we define four flags on V called $[0]$, $[D]$, $[0^*]$, $[D^*]$; for $0 \leqslant i \leqslant d$ the ith component of $[0]$ (resp. $[D]$, $[0^*]$, $[D^*]$) is $\mathrm{Span}\{w_0, w_1, \ldots, w_i\}$ (resp. $\mathrm{Span}\{w_d, w_{d-1}, \ldots, w_{d-i}\}$, $\mathrm{Span}\{v_0, v_1, \ldots, v_i\}$, $\mathrm{Span}\{v_d, v_{d-1}, \ldots, v_{d-i}\}$). These four flags are mutually opposite in the sense of [31, Theorem 7.3]. We show that up to multiplication by a nonzero scalar, S is the unique linear transformation on V that fixes each of $[0]$, $[D]$ and sends $[0^*]$ to $[D^*]$. A *decomposition* of V is a sequence of one-dimensional subspaces whose direct sum is V. Let x, y denote an ordered pair of distinct elements from the set $\{0, D, 0^*, D^*\}$. By [31, Theorem 8.3] there exists a decomposition $[xy]$ of V such that for $0 \leqslant i \leqslant d$ the ith component of $[xy]$ is the intersection of the ith component of $[x]$ and the $(d-i)$th component of $[y]$. We show that up to multiplication by a nonzero scalar, S is the unique linear transformation on V that sends $[0^*0]$ to $[D^*0]$ and

$[0^*D]$ to $[D^*D]$. By our earlier remarks there exists a unique linear transformation $S^*: V \to V$ that sends w_0 to a scalar multiple of w_d, fixes v_0, and sends v_i to a scalar multiple of v_i for $1 \leqslant i \leqslant d$. We show that each component of $[0^*D]$ (resp. $[D^*D]$, $[0^*0]$, $[D^*0]$) is an eigenspace for $S^*S^{-1}S^{*-1}S$ (resp. $S^*SS^{*-1}S^{-1}$, $S^{*-1}S^{-1}S^*S$, $S^{*-1}SS^*S^{-1}$). We find the corresponding eigenvalues. We consider a certain basis for V whose ith component is contained in the ith component of $[0^*D]$ for $0 \leqslant i \leqslant d$. With respect to this basis the matrix representing A (resp. A^*) is lower bidiagonal (resp. upper bidiagonal) [29, Lemma 3.9]. We display the matrices that represent S and S^* with respect to this basis. In a related result we characterize the Leonard pair concept in terms of the switching element. We finish the paper with some open problems.

2. Leonard systems

When working with a Leonard pair, it is convenient to consider a closely related object called a *Leonard system*. To prepare for our definition of a Leonard system, we recall a few concepts from linear algebra. Let d denote a nonnegative integer and let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices that have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. We let $\mathbb{K}^{d+1}$ denote the $\mathbb{K}$-vector space of all $d+1$ by 1 matrices that have entries in $\mathbb{K}$. We index the rows by $0, 1, \ldots, d$. We view $\mathbb{K}^{d+1}$ as a left module for $\mathrm{Mat}_{d+1}(\mathbb{K})$. We observe this module is irreducible. For the rest of this paper, let $\mathscr{A}$ denote a $\mathbb{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$ and let V denote a simple left $\mathscr{A}$-module. We remark that V is unique up to isomorphism of $\mathscr{A}$-modules, and that V has dimension $d+1$. Let $\{v_i\}_{i=0}^d$ denote a basis for V. For $X \in \mathscr{A}$ and $Y \in \mathrm{Mat}_{d+1}(\mathbb{K})$, we say *$Y$ represents X with respect to $\{v_i\}_{i=0}^d$* whenever $Xv_j = \sum_{i=0}^d Y_{ij}v_i$ for $0 \leqslant j \leqslant d$. For $A \in \mathscr{A}$ we say A is *multiplicity-free* whenever it has $d+1$ mutually distinct eigenvalues in $\mathbb{K}$. Assume A is multiplicity-free. Let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of A, and for $0 \leqslant i \leqslant d$ put

$$E_i = \prod_{\substack{0 \leqslant j \leqslant d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j}, \tag{1}$$

where I denotes the identity of $\mathscr{A}$. We observe (i) $AE_i = \theta_i E_i$ $(0 \leqslant i \leqslant d)$; (ii) $E_iE_j = \delta_{i,j}E_i$ $(0 \leqslant i, j \leqslant d)$; (iii) $\sum_{i=0}^d E_i = I$; (iv) $A = \sum_{i=0}^d \theta_i E_i$. Let $\mathscr{D}$ denote the subalgebra of $\mathscr{A}$ generated by A. Using (i)–(iv) we find the sequence $\{E_i\}_{i=0}^d$ is a basis for the $\mathbb{K}$-vector space $\mathscr{D}$. We call E_i the *primitive idempotent* of A associated with θ_i. It is helpful to think of these primitive idempotents as follows. Observe

$$V = E_0V + E_1V + \cdots + E_dV \quad \text{(direct sum)}.$$

For $0 \leqslant i \leqslant d$, E_iV is the (one-dimensional) eigenspace of A in V associated with the eigenvalue θ_i, and E_i acts on V as the projection onto this eigenspace. We note that for $X \in \mathscr{A}$ the following are equivalent: (i) $X \in \mathscr{D}$; (ii) $XA = AX$; (iii) $XE_iV \subseteq E_iV$ for $0 \leqslant i \leqslant d$.

By a *Leonard pair in $\mathscr{A}$* we mean an ordered pair of elements taken from $\mathscr{A}$ that act on V as a Leonard pair in the sense of Definition 1.1. We now define a Leonard system.

Definition 2.1 [29]. By a *Leonard system* in $\mathscr{A}$ we mean a sequence

$$\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$$

that satisfies (i)–(v) below.

(i) Each of A, A^* is a multiplicity-free element in $\mathscr{A}$.
(ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A.

(iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^*.
(iv) For $0 \leqslant i, j \leqslant d$

$$E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1. \end{cases} \tag{2}$$

(v) For $0 \leqslant i, j \leqslant d$

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1. \end{cases} \tag{3}$$

We say Φ *is over* $\mathbb{K}$.

Leonard systems are related to Leonard pairs as follows. Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathscr{A}$. Then A, A^* is a Leonard pair in $\mathscr{A}$ [37, Section 3]. Conversely, suppose A, A^* is a Leonard pair in $\mathscr{A}$. Then each of A, A^* is multiplicity-free [29, Lemma 1.3]. Moreover, there exists an ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents of A, and there exists an ordering $\{E_i^*\}_{i=0}^d$ of the primitive idempotents of A^*, such that $\left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ is a Leonard system in $\mathscr{A}$ [37, Lemma 3.3].

3. The D_4 action

For a given Leonard system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ in $\mathscr{A}$, each of the following is a Leonard system in $\mathscr{A}$:

$$\Phi^* := (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d),$$
$$\Phi^\downarrow := (A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d),$$
$$\Phi^\Downarrow := (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).$$

Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all Leonard systems in $\mathscr{A}$

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \tag{4}$$
$$\Downarrow * = * \downarrow, \quad \downarrow * = * \Downarrow, \quad \downarrow\Downarrow = \Downarrow\downarrow . \tag{5}$$

The group generated by symbols $*, \downarrow, \Downarrow$ subject to relations (4) and (5) is the dihedral group D_4. We recall that D_4 is the group of symmetries of a square and has 8 elements. Apparently $*, \downarrow, \Downarrow$ induce an action of D_4 on the set of all Leonard systems in $\mathscr{A}$. Two Leonard systems will be called *relatives* whenever they are in the same orbit of this D_4 action. The relatives of Φ are as follows:

Name	Relative
Φ	$\left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$
$\Phi^\downarrow$	$\left(A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d\right)$
$\Phi^\Downarrow$	$\left(A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$
$\Phi^{\downarrow\Downarrow}$	$\left(A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d\right)$
Φ^*	$\left(A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d\right)$
$\Phi^{\downarrow *}$	$\left(A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d\right)$
$\Phi^{\Downarrow *}$	$\left(A^*; \{E_i^*\}_{i=0}^d; A; \{E_{d-i}\}_{i=0}^d\right)$
$\Phi^{\downarrow\Downarrow *}$	$\left(A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_{d-i}\}_{i=0}^d\right)$

We will use the following notational convention.

Definition 3.1. For $g \in D_4$ and for an object f associated with Φ we let f^g denote the corresponding object associated with $\Phi^{g^{-1}}$.

4. The parameter array

In this section, we recall some parameters.

Definition 4.1. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathscr{A}$. For $0 \leqslant i \leqslant d$ we let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). We refer to $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) as the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ. We observe $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct and contained in $\mathbb{K}$.

We will use the following notation. Let λ denote an indeterminate and let $\mathbb{K}[\lambda]$ denote the $\mathbb{K}$-algebra consisting of all polynomials in λ that have coefficients in $\mathbb{K}$.

Definition 4.2. Let $\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system in $\mathscr{A}$. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denote the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ. For $0 \leqslant i \leqslant d$ we define polynomials $\tau_i, \eta_i, \tau_i^*, \eta_i^*$ in $\mathbb{K}[\lambda]$ as follows:

$$\begin{aligned}
\tau_i &= (\lambda - \theta_0)(\lambda - \theta_1)\cdots(\lambda - \theta_{i-1}),\\
\eta_i &= (\lambda - \theta_d)(\lambda - \theta_{d-1})\cdots(\lambda - \theta_{d-i+1}),\\
\tau_i^* &= (\lambda - \theta_0^*)(\lambda - \theta_1^*)\cdots(\lambda - \theta_{i-1}^*),\\
\eta_i^* &= (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*)\cdots(\lambda - \theta_{d-i+1}^*).
\end{aligned}$$

Note that each of $\tau_i, \eta_i, \tau_i^*, \eta_i^*$ is monic with degree i for $0 \leqslant i \leqslant d$.

Definition 4.3 [20, *Theorem* 4.6]. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathscr{A}$. Referring to Definition 4.2, we define scalars

$$\varphi_i = (\theta_0^* - \theta_i^*)\frac{\operatorname{tr}(\tau_i(A)E_0^*)}{\operatorname{tr}(\tau_{i-1}(A)E_0^*)} \quad (1 \leqslant i \leqslant d), \tag{6}$$

$$\phi_i = (\theta_0^* - \theta_i^*)\frac{\operatorname{tr}(\eta_i(A)E_0^*)}{\operatorname{tr}(\eta_{i-1}(A)E_0^*)} \quad (1 \leqslant i \leqslant d), \tag{7}$$

where tr means trace. We note that in (6) and (7) the denominators are nonzero by [20, Corollary 4.5]. The sequence $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) is called the *first split sequence* (resp. *second split sequence*) of Φ.

Definition 4.4. Let $\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system in $\mathscr{A}$. By the *parameter array of* Φ we mean the sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$, where the θ_i, θ_i^* are from Definition 4.1 and the φ_i, ϕ_i are from Definition 4.3.

Theorem 4.5 [29, Theorem 1.9]. *Let d denote a nonnegative integer and let*

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d) \tag{8}$$

denote a sequence of scalars taken from $\mathbb{K}$. Then there exists a Leonard system Φ over $\mathbb{K}$ with parameter array (8) *if and only if* (PA1)–(PA5) *hold below*:

(PA1) $\varphi_i \neq 0, \phi_i \neq 0 \ (1 \leqslant i \leqslant d)$.
(PA2) $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*$ *if* $i \neq j$ $(0 \leqslant i, j \leqslant d)$.
(PA3) *For* $1 \leqslant i \leqslant d$

$$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d).$$

(PA4) *For* $1 \leqslant i \leqslant d$

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0).$$

(PA5) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \tag{9}$$

are equal and independent of i for $2 \leqslant i \leqslant d - 1$.

Suppose (PA1)–(PA5) *hold. Then Φ is unique up to isomorphism of Leonard systems*.

The D_4 action affects the parameter array as follows.

Lemma 4.6 [29, Theorem 1.11]. *Let $\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system in $\mathscr{A}$ and let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$ denote the parameter array of Φ. Then the following* (i)–(iii) *hold*:

(i) *The parameter array of Φ^* is*

$$\left(\{\theta_i^*\}_{i=0}^d; \{\theta_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d\right).$$

(ii) *The parameter array of $\Phi^\downarrow$ is*

$$\left(\{\theta_i\}_{i=0}^d; \{\theta_{d-i}^*\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d\right).$$

(iii) *The parameter array of $\Phi^\Downarrow$ is*

$$\left(\{\theta_{d-i}\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d\right).$$

We finish this section with a comment.

Lemma 4.7. *Let $\left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system in $\mathscr{A}$. Let $\mathscr{D}$ denote the subalgebra of $\mathscr{A}$ generated by A, and let X denote an element of $\mathscr{D}$ such that $XE_0^* = 0$. Then $X = 0$.*

Proof. Immediate from [37, Lemma 5.9]. $\square$

5. The switching element S

Definition 5.1. For a Leonard system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ in $\mathscr{A}$ we define

$$S = \sum_{r=0}^{d} \frac{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}}{\varphi_1 \varphi_2 \cdots \varphi_r} E_r, \tag{10}$$

where $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) denotes the first (resp. second) split sequence of Φ. We call S the *switching element* for Φ.

Note 5.2. Let $\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system. In what follows we will often make use of the switching element of Φ^*. By (10) and Lemma 4.6(i):

$$S^* = \sum_{r=0}^{d} \frac{\phi_1 \phi_2 \cdots \phi_r}{\varphi_1 \varphi_2 \cdots \varphi_r} E_r^*. \tag{11}$$

We call S^* the *dual switching element* for Φ.

Lemma 5.3. *The switching element (10) and the dual switching element (11) are invertible with*

$$S^{-1} = \sum_{r=0}^{d} \frac{\varphi_1 \varphi_2 \cdots \varphi_r}{\phi_d \phi_{d-1} \cdots \phi_{d-r+1}} E_r, \tag{12}$$

$$S^{*-1} = \sum_{r=0}^{d} \frac{\varphi_1 \varphi_2 \cdots \varphi_r}{\phi_1 \phi_2 \cdots \phi_r} E_r^*. \tag{13}$$

Proof. To obtain (12) we note that the sum on the right in (10) times the sum on the right in (12) is equal to the identity; this is verified using equations (ii), (iii) below (1). Line (13) is similarly obtained. □

Theorem 5.4. *Let $\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system with switching element S and dual switching element S^*. Then the switching element and the dual switching element for the relatives of Φ are given in the following table:*

Relative	Φ	$\Phi^{\downarrow}$	$\Phi^{\Downarrow}$	$\Phi^{\downarrow\Downarrow}$	Φ^*	$\Phi^{\downarrow *}$	$\Phi^{\Downarrow *}$	$\Phi^{\downarrow\Downarrow *}$
Switching element	S	S^{-1}	$\varphi\phi^{-1}S$	$\varphi^{-1}\phi S^{-1}$	S^*	$\varphi\phi^{-1}S^*$	S^{*-1}	$\varphi^{-1}\phi S^{*-1}$
Dual switching element	S^*	$\varphi\phi^{-1}S^*$	S^{*-1}	$\varphi^{-1}\phi S^{*-1}$	S	S^{-1}	$\varphi\phi^{-1}S$	$\varphi^{-1}\phi S^{-1}$

In the above table we abbreviate

$$\varphi = \varphi_1 \varphi_2 \cdots \varphi_d, \quad \phi = \phi_1 \phi_2 \cdots \phi_d,$$

where $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) is the first (resp. second) split sequence of Φ.

Proof. Apply D_4 to (10) and use Lemma 4.6. □

We now describe the switching element from various points of view.

6. Representing S as a polynomial

Let $\big(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\big)$ denote a Leonard system in $\mathcal{A}$ with switching element S. Let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by A, and recall $\{E_i\}_{i=0}^d$ is a basis for $\mathcal{D}$. By this and (10) we find $S \in \mathcal{D}$, so S is a polynomial in A. In the present section, we find this polynomial.

Lemma 6.1. *Let $\big(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\big)$ denote a Leonard system in $\mathcal{A}$. Then for $0 \leqslant i \leqslant d$ there exists a unique monic polynomial p_i in $\mathbb{K}[\lambda]$ with degree i such that*

$$p_i(A)E_0^*V = E_i^*V.$$

Proof. The existence of p_i is established in [37, Theorem 8.3]. Concerning uniqueness, suppose we are given a monic polynomial p_i' in $\mathbb{K}[\lambda]$ of degree i such that $p_i'(A)E_0^*V = E_i^*V$. We show $p_i = p_i'$. To this end we define $f = p_i - p_i'$ and show $f = 0$. By construction $f(A)E_0^*V \subseteq E_i^*V$. Each of p_i, p_i' is monic of degree i so the degree of f is at most $i-1$. By this and (3) we find $f(A)E_0^*V$ is included in $\sum_{k=0}^{i-1} E_k^*V$. By these comments we find $f(A)E_0^*V = 0$ so $f(A)E_0^* = 0$. Now $f(A) = 0$ in view of Lemma 4.7. This implies $f = 0$ since $I, A, A^2, \ldots, A^d$ are linearly independent. $\square$

Lemma 6.2 [37, Lemma 17.5]. *Let $\big(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\big)$ denote a Leonard system and let $\big(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\big)$ denote the corresponding parameter array. Let the polynomials $\{p_i\}_{i=0}^d$ be from Lemma 6.1. Then*

$$p_i(\theta_0) = \frac{\varphi_1\varphi_2\cdots\varphi_i}{\tau_i^*(\theta_i^*)} \quad (0 \leqslant i \leqslant d). \tag{14}$$

Moreover, $p_i(\theta_0) \neq 0$.

Definition 6.3 [37, *Definition* 14.1]. Let $\big(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\big)$ denote a Leonard system over $\mathbb{K}$ and let the polynomials $\{p_i\}_{i=0}^d$ be as in Lemma 6.1. For $0 \leqslant i \leqslant d$ we define

$$u_i = \frac{p_i}{p_i(\theta_0)}, \tag{15}$$

where θ_0 is from Definition 4.1.

Lemma 6.4. *Let $\big(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\big)$ denote a Leonard system in $\mathcal{A}$ and let the polynomials $\{u_i\}_{i=0}^d$ be from Definition 6.3. Then*

$$u_i(A)E_0^*V = E_i^*V \quad (0 \leqslant i \leqslant d).$$

Proof. Combine Lemma 6.1 and (15). $\square$

Lemma 6.5 [37, Theorem 14.7]. *Let $\big(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\big)$ denote a Leonard system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let the polynomials $\{u_i\}_{i=0}^d$ be as in Definition 6.3 and recall $\{u_i^*\}_{i=0}^d$ are the corresponding polynomials for Φ^*. Then for $0 \leqslant i, j \leqslant d$*

$$u_i(\theta_j) = u_j^*(\theta_i^*). \tag{16}$$

Theorem 6.6. *Let $\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system and let S denote the switching element for Φ. Then*

$$S = u_d(A), \tag{17}$$

where the polynomial u_d is from Definition 6.3.

Proof. By D_4 symmetry it suffices to show $S^* = u_d^*(A^*)$. Using the comments below (1) we find

$$u_d^*(A^*) = \sum_{i=0}^{d} u_d^*(\theta_i^*) E_i^*. \tag{18}$$

For $0 \leqslant i \leqslant d$ we compute $u_d^*(\theta_i^*)$ as follows. By Lemma 6.1 the polynomial p_i is invariant under $\Downarrow$; that is $p_i^{\Downarrow} = p_i$. We apply $\Downarrow$ to (14) using this and Lemma 4.6 to get

$$p_i(\theta_d) = \frac{\phi_1\phi_2\cdots\phi_i}{\tau_i^*(\theta_i^*)}.$$

Combining this with (14) and (15) we find

$$u_i(\theta_d) = \frac{\phi_1\phi_2\cdots\phi_i}{\varphi_1\varphi_2\cdots\varphi_i}.$$

By this and Lemma 6.5 we get

$$u_d^*(\theta_i^*) = \frac{\phi_1\phi_2\cdots\phi_i}{\varphi_1\varphi_2\cdots\varphi_i}. \tag{19}$$

Evaluating (18) using (19) and comparing the result with (11) we find $S^* = u_d^*(A^*)$. The result follows. □

The switching element is characterized as follows.

Theorem 6.7. *Let $\left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system in $\mathscr{A}$ with switching element S. Let $\mathscr{D}$ denote the subalgebra of $\mathscr{A}$ generated by A. Then for all nonzero $X \in \mathscr{A}$ the following* (i), (ii) *are equivalent*:

(i) *X is a scalar multiple of S.*
(ii) *$X \in \mathscr{D}$ and $XE_0^*V \subseteq E_d^*V$.*

Suppose (i), (ii) *hold. Then $XE_0^*V = E_d^*V$.*

Proof. (i) ⇒ (ii): We mentioned in the first paragraph of this section that $S \in \mathscr{D}$. We have $SE_0^*V = E_d^*V$ by Lemma 6.4 and Theorem 6.6.

(ii) ⇒ (i): For $0 \leqslant i \leqslant d$ we define $\mathscr{D}_i = \text{Span}\{u_i(A)\}$. Observe that $\mathscr{D} = \sum_{i=0}^d \mathscr{D}_i$ (direct sum). Also observe by Lemma 6.4 that $\mathscr{D}_i E_0^*V = E_i^*V$ for $0 \leqslant i \leqslant d$. We assume $XE_0^*V \subseteq E_d^*V$ so $X \in \mathscr{D}_d$ and in other words X is a scalar multiple of $u_d(A)$. By this and Theorem 6.6 we find X is a scalar multiple of S.

Now suppose (i), (ii) hold. We mentioned in the proof of (i) ⇒ (ii) that $SE_0^*V = E_d^*V$. But X is nonzero and a scalar multiple of S so $XE_0^*V = E_d^*V$. □

7. Decompositions and flags

In this section, we recall the notion of a decomposition and a flag.

By a *decomposition* of V we mean a sequence $\{V_i\}_{i=0}^d$ of subspaces of V such that V_i has dimension 1 for $0 \leqslant i \leqslant d$ and $\sum_{i=0}^d V_i = V$ (direct sum). Let $\{V_i\}_{i=0}^d$ denote a decomposition of V. By the *inversion* of this decomposition we mean the decomposition $\{V_{d-i}\}_{i=0}^d$.

By a *flag* on V we mean a sequence $\{F_i\}_{i=0}^d$ of subspaces of V such that F_i has dimension $i+1$ for $0 \leqslant i \leqslant d$ and $F_{i-1} \subseteq F_i$ for $1 \leqslant i \leqslant d$. The following construction yields a flag on V. Let $\{V_i\}_{i=0}^d$ denote a decomposition of V. Define

$$F_i = V_0 + V_1 + \cdots + V_i \quad (0 \leqslant i \leqslant d).$$

Then $\{F_i\}_{i=0}^d$ is a flag on V. We say this flag is *induced* by the decomposition $\{V_i\}_{i=0}^d$.

We recall what it means for two flags on V to be *opposite*. Suppose we are given two flags on V: $\{F_i\}_{i=0}^d$ and $\{F'_i\}_{i=0}^d$. We say these flags are *opposite* whenever there exists a decomposition $\{V_i\}_{i=0}^d$ of V such that

$$F_i = V_0 + V_1 + \cdots + V_i, \quad F'_i = V_d + V_{d-1} + \cdots + V_{d-i}$$

for $0 \leqslant i \leqslant d$. In this case

$$F_i \cap F_j = 0 \quad \text{if } i + j < d \qquad (0 \leqslant i, j \leqslant d)$$

and

$$V_i = F_i \cap F'_{d-i} \quad (0 \leqslant i \leqslant d).$$

In particular the decomposition $\{V_i\}_{i=0}^d$ is uniquely determined by the given flags. We say this decomposition is *induced* by the given flags.

We end this section with some notation.

Notation 7.1. Let $F = \{F_i\}_{i=0}^d$ denote a sequence of subspaces of V. Then for $X \in \mathscr{A}$ we write XF to denote the sequence $\{XF_i\}_{i=0}^d$. We say X *fixes* F whenever $XF = F$. Let $F' = \{F'_i\}_{i=0}^d$ denote a second sequence of subspaces of V. We write $F \subseteq F'$ whenever $F_i \subseteq F'_i$ for $0 \leqslant i \leqslant d$.

8. Some decompositions and flags associated with a Leonard system

We now return our attention to Leonard systems. Let $\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d\right)$ denote a Leonard system in $\mathscr{A}$. Using Φ we construct four mutually opposite flags and consider the decompositions that they induce. We start with a definition.

Definition 8.1. For notational convenience let Ω denote the set consisting of four symbols 0, D, 0^*, D^*.

Definition 8.2. Let $\left(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d\right)$ denote a Leonard system in $\mathscr{A}$. For $z \in \Omega$ we define a flag on V which we denote by $[z]$. To define this flag we display the ith component for $0 \leqslant i \leqslant d$.

z	ith component of $[z]$
0	$E_0V + E_1V + \cdots + E_iV$
D	$E_dV + E_{d-1}V + \cdots + E_{d-i}V$
0^*	$E^*_0V + E^*_1V + \cdots + E^*_iV$
D^*	$E^*_dV + E^*_{d-1}V + \cdots + E^*_{d-i}V$

Lemma 8.3. *Referring to Definition* 8.2, *the following* (i)–(iv) *hold for* $0 \leqslant i \leqslant d$:

(i) *The* i*th component of* $[0]$ *is equal to* $\eta_{d-i}(A)V$.
(ii) *The* i*th component of* $[D]$ *is equal to* $\tau_{d-i}(A)V$.
(iii) *The* i*th component of* $[0^*]$ *is equal to* $\eta^*_{d-i}(A^*)V$.
(iv) *The* i*th component of* $[D^*]$ *is equal to* $\tau^*_{d-i}(A^*)V$.

Proof. (i): Recall that $V = \sum_{j=0}^{d} E_jV$ (direct sum). Further recall that for $0 \leqslant j \leqslant d$, E_jV is an eigenspace for A with eigenvalue θ_j. This implies that for $0 \leqslant j, k \leqslant d$, $(A - \theta_k I)E_jV$ equals 0 if $j = k$ and E_jV if $j \neq k$. By these comments and Definition 4.2 we have $\eta_{d-i}(A)E_jV = 0$ for $i + 1 \leqslant j \leqslant d$ and $\eta_{d-i}(A)E_jV = E_jV$ for $0 \leqslant j \leqslant i$. Therefore, $\eta_{d-i}(A)V = \sum_{j=0}^{i} E_jV$ and this is the ith component of $[0]$.

(ii)–(iv): Similar. □

Lemma 8.4 [31, Theorem 7.3]. *The four flags in Definition* 8.2 *are mutually opposite*.

Definition 8.5. Let $\Phi = \left(A; \{E_i\}_{i=0}^{d}; A^*; \{E^*_i\}_{i=0}^{d}\right)$ denote a Leonard system in $\mathscr{A}$. Let z, w denote an ordered pair of distinct elements of Ω. By Lemma 8.4 the flags $[z]$, $[w]$ are opposite. Let $[zw]$ denote the decomposition of V induced by $[z]$, $[w]$.

We mention a few basic properties of the decompositions from Definition 8.5.

Lemma 8.6. *Referring to Definition* 8.5, *for distinct* $z, w \in \Omega$ *the following* (i)–(iii) *hold*:

(i) *The decomposition* $[zw]$ *is the inversion of* $[wz]$.
(ii) *For* $0 \leqslant i \leqslant d$ *the* i*th component of* $[zw]$ *is the intersection of the* i*th component of* $[z]$ *and the* $(d - i)$*th component of* $[w]$.
(iii) *The decomposition* $[zw]$ *induces* $[z]$ *and the inversion of* $[zw]$ *induces* $[w]$.

Proof. Routine using Section 7 and Definition 8.5. □

Example 8.7. We display some of the decompositions from Definition 8.5. For each decomposition in the table below we give the ith component for $0 \leqslant i \leqslant d$:

Decomposition	ith component
$[0^*D]$	$(E^*_0V + \cdots + E^*_iV) \cap (E_iV + \cdots + E_dV)$
$[D^*D]$	$(E^*_dV + \cdots + E^*_{d-i}V) \cap (E_iV + \cdots + E_dV)$
$[0^*0]$	$(E^*_0V + \cdots + E^*_iV) \cap (E_{d-i}V + \cdots + E_0V)$
$[D^*0]$	$(E^*_dV + \cdots + E^*_{d-i}V) \cap (E_{d-i}V + \cdots + E_0V)$
$[0D]$	E_iV
$[0^*D^*]$	E^*_iV

Lemma 8.8. *Referring to Definition* 8.5, *the following* (i)–(iv) *hold for* $0 \leqslant i \leqslant d$:

(i) *The* ith *component of* $[0^*D]$ *is equal to* $\tau_i(A)E_0^*V$ *and* $\eta_{d-i}^*(A^*)E_dV$.
(ii) *The* ith *component of* $[D^*D]$ *is equal to* $\tau_i(A)E_d^*V$ *and* $\tau_{d-i}^*(A^*)E_dV$.
(iii) *The* ith *component of* $[0^*0]$ *is equal to* $\eta_i(A)E_0^*V$ *and* $\eta_{d-i}^*(A^*)E_0V$.
(iv) *The* ith *component of* $[D^*0]$ *is equal to* $\eta_i(A)E_d^*V$ *and* $\tau_{d-i}^*(A^*)E_0V$.

Proof. (i): We first show that $\tau_i(A)E_0^*V$ is equal to the ith component of $[0^*D]$. Denote this ith component by U_i. By Lemma 8.3(ii) $\tau_i(A)V$ is equal to $\sum_{j=i}^d E_jV$, so $\tau_i(A)E_0^*V$ is contained in $\sum_{j=i}^d E_jV$. By (3) and since $\tau_i(A)$ has degree i we find $\tau_i(A)E_0^*V$ is contained in $\sum_{k=0}^i E_k^*V$. By these comments and the definition of U_i we find $\tau_i(A)E_0^*V \subseteq U_i$. By Lemma 4.7 we have $\tau_i(A)E_0^* \neq 0$ so $\tau_i(A)E_0^*V \neq 0$. By this and since U_i has dimension 1 we find $\tau_i(A)E_0^*V = U_i$. We have now shown that $\tau_i(A)E_0^*V$ is equal to the ith component of $[0^*D]$. In a similar way we find that $\eta_{d-i}^*(A^*)E_dV$ is equal to the ith component of $[0^*D]$.

(ii)–(iv): Apply (i) to the relatives of Φ. □

9. The action of S on the flags

Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathscr{A}$ and let S denote the corresponding switching element. In this section, we characterize S via its action on the flags from Definition 8.2.

Theorem 9.1. *Let* $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ *denote a Leonard system in* $\mathscr{A}$ *and let* S *denote the corresponding switching element. Then for all nonzero* $X \in \mathscr{A}$ *the following* (i), (ii) *are equivalent*:

(i) X *is a scalar multiple of* S.
(ii) $X[0] \subseteq [0]$, $X[D] \subseteq [D]$, *and* $X[0^*] \subseteq [D^*]$.

Suppose (i), (ii) *hold. Then equality is attained everywhere in* (ii).

Proof. (i) ⇒ (ii): We show $S[0] = [0]$, $S[D] = [D]$, and $S[0^*] = [D^*]$. For $0 \leqslant i \leqslant d$ we find $SE_iV \subseteq E_iV$ since S is a polynomial in A, and $SE_iV = E_iV$ since S^{-1} exists. Therefore, $S[0] = [0]$ and $S[D] = [D]$. We now show that $S[0^*] = [D^*]$. To this end we fix an integer i $(0 \leqslant i \leqslant d)$ and show

$$S(E_0^*V + E_1^*V + \cdots + E_i^*V) = E_d^*V + E_{d-1}^*V + \cdots + E_{d-i}^*V. \tag{20}$$

Using Lemma 6.4 and Theorem 6.6 we find that for $0 \leqslant j \leqslant i$

$$\begin{aligned} SE_j^*V &= u_d(A)u_j(A)E_0^*V \\ &= u_j(A)u_d(A)E_0^*V \\ &= u_j(A)E_d^*V. \end{aligned}$$

By (3) and since the polynomial u_j has degree j

$$u_j(A)E_d^*V \subseteq E_{d-j}^*V + E_{d-j+1}^*V + \cdots + E_d^*V.$$

Combining these comments we obtain

$$SE_j^*V \subseteq E_{d-j}^*V + E_{d-j+1}^*V + \cdots + E_d^*V$$

and it follows that

$$S(E_0^*V + E_1^*V + \cdots + E_i^*V) \subseteq E_d^*V + E_{d-1}^*V + \cdots + E_{d-i}^*V.$$

In the above inclusion each side has the same dimension since S^{-1} exists, so the inclusion becomes equality and (20) holds.

(ii) ⇒ (i): By Theorem 6.7 it suffices to show that $X \in \mathscr{D}$ and $XE_0^*V \subseteq E_d^*V$. We first show $X \in \mathscr{D}$. Recall that the flags $[0]$, $[D]$ induce the decomposition $[0D]$. We assume $X[0] \subseteq [0]$ and $X[D] \subseteq [D]$ so $X[0D] \subseteq [0D]$. This means that $XE_iV \subseteq E_iV$ for $0 \leqslant i \leqslant d$, so $X \in \mathscr{D}$. To get $XE_0^*V \subseteq E_d^*V$, consider the 0th component in the inclusion $X[0^*] \subseteq [D^*]$.

Suppose (i), (ii) hold. We mentioned in the proof of (i) ⇒ (ii) that $S[0] = [0]$, $S[D] = [D]$, and $S[0^*] = [D^*]$. But X is nonzero and a scalar multiple of S so $X[0] = [0]$, $X[D] = [D]$, and $X[0^*] = [D^*]$. □

10. The action of S on the decompositions

Let $\left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system in $\mathscr{A}$ and let S denote the corresponding switching element. In this section, we characterize S via its action on the decompositions from Definition 8.5.

Theorem 10.1. *Let* $\left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ *denote a Leonard system in* $\mathscr{A}$ *and let* S *denote the corresponding switching element. Then for all nonzero* $X \in \mathscr{A}$ *the following* (i), (ii) *are equivalent*:

(i) *X is a scalar multiple of S.*
(ii) $X[0^*0] \subseteq [D^*0]$ *and* $X[0^*D] \subseteq [D^*D]$.

Suppose (i), (ii) *hold. Then equality holds everywhere in* (ii).

Proof. (i) ⇒ (ii): By Theorem 9.1 we have $S[0^*] \subseteq [D^*]$ and $S[0] \subseteq [0]$ so $S[0^*0] \subseteq [D^*0]$. Since each component of a decomposition has dimension 1 and since S^{-1} exists, we find $S[0^*0] = [D^*0]$. In a similar way we obtain $S[0^*D] = [D^*D]$.

(ii) ⇒ (i): By Theorem 9.1 it suffices to show $X[0] \subseteq [0]$, $X[D] \subseteq [D]$, and $X[0^*] \subseteq [D^*]$. We first show $X[0] \subseteq [0]$. By Lemma 8.6(i) and since $X[0^*0] \subseteq [D^*0]$ we find $X[00^*] \subseteq [0D^*]$. The decompositions $[00^*]$ and $[0D^*]$ each induce the flag $[0]$ by Lemma 8.6(iii) so $X[0] \subseteq [0]$. Next we show $X[D] \subseteq [D]$. By Lemma 8.6(i) and since $X[0^*D] \subseteq [D^*D]$ we find $X[D0^*] \subseteq [DD^*]$. The decompositions $[D0^*]$ and $[DD^*]$ each induce the flag $[D]$ by Lemma 8.6(iii) so $X[D] \subseteq [D]$. Finally, we show $X[0^*] \subseteq [D^*]$. By Lemma 8.6(iii) we find that $[0^*D]$ induces $[0^*]$ and $[D^*D]$ induces $[D^*]$. By this and since $X[0^*D] \subseteq [D^*D]$ we find $X[0^*] \subseteq [D^*]$.

Suppose (i), (ii) hold. We mentioned in the proof of (i) ⇒ (ii) that $S[0^*0] = [D^*0]$ and $S[0^*D] = [D^*D]$. But X is nonzero and a scalar multiple of S so $X[0^*0] = [D^*0]$ and $X[0^*D] = [D^*D]$. □

11. Some group commutators

Let $\left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system with switching element S and dual switching element S^*. In this section, we consider linear transformations such as $S^*S^{-1}S^{*-1}S$. As we will see, these maps are closely related to the decompositions from Definition 8.5. We start with a lemma.

Lemma 11.1. *Let $\left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system in $\mathcal{A}$, with switching element S and dual switching element S^*. Then referring to Definition 8.2 the following* (i)–(iv) *hold*:

(i) $S^*S^{-1}S^{*-1}S$ *fixes each of* $[0^*]$, $[D]$.
(ii) $S^*SS^{*-1}S^{-1}$ *fixes each of* $[D^*]$, $[D]$.
(iii) $S^{*-1}S^{-1}S^*S$ *fixes each of* $[0^*]$, $[0]$.
(iv) $S^{*-1}SS^*S^{-1}$ *fixes each of* $[D^*]$, $[0]$.

Proof. By Theorem 9.1 we find $S[0] = [0]$, $S[D] = [D]$, and $S[0^*] = [D^*]$. Applying this to Φ^* we find $S^*[0^*] = [0^*]$, $S^*[D^*] = [D^*]$, and $S^*[0] = [D]$. Combining these comments we routinely obtain the result. □

Corollary 11.2. *Let $\left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system in $\mathcal{A}$, with switching element S and dual switching element S^*. Then referring to Definition 8.5 the following* (i)–(iv) *hold*:

(i) $S^*S^{-1}S^{*-1}S$ *fixes* $[0^*D]$.
(ii) $S^*SS^{*-1}S^{-1}$ *fixes* $[D^*D]$.
(iii) $S^{*-1}S^{-1}S^*S$ *fixes* $[0^*0]$.
(iv) $S^{*-1}SS^*S^{-1}$ *fixes* $[D^*0]$.

Proof. (i): For notational convenience abbreviate $T = S^*S^{-1}S^{*-1}S$. By Lemma 11.1 we have $T[0^*] = [0^*]$ and $T[D] = [D]$ so $T[0^*D] \subseteq [0^*D]$. By this and since T^{-1} exists we find $T[0^*D] = [0^*D]$.

(ii)–(iv) Apply (i) to the relatives of Φ and use Theorem 5.4. □

Referring to Corollary 11.2, each part (i)–(iv) is asserting that for $0 \leqslant i \leqslant d$, the ith component of the given decomposition is an eigenspace for the given operator. We now find the corresponding eigenvalue. We will focus on case (i); the eigenvalues for the remaining cases will be found using the D_4 action.

Lemma 11.3 [22, Theorem 5.2]. *Let $\left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system and let $\left(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\right)$ denote the corresponding parameter array. Then for $0 \leqslant i \leqslant d$*

$$\eta_i(A)E_0^*E_0 = \frac{\phi_1\phi_2\cdots\phi_i}{\eta_d^*(\theta_0^*)}\eta_{d-i}^*(A^*)E_0, \tag{21}$$

$$\eta_i(A)E^*_dE_0 = \frac{\varphi_d\varphi_{d-1}\cdots\varphi_{d-i+1}}{\tau^*_d(\theta^*_d)}\tau^*_{d-i}(A^*)E_0, \tag{22}$$

$$\tau_i(A)E^*_0E_d = \frac{\varphi_1\varphi_2\cdots\varphi_i}{\eta^*_d(\theta^*_0)}\eta^*_{d-i}(A^*)E_d, \tag{23}$$

$$\tau_i(A)E^*_dE_d = \frac{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}}{\tau^*_d(\theta^*_d)}\tau^*_{d-i}(A^*)E_d, \tag{24}$$

$$\eta^*_i(A^*)E_0E^*_0 = \frac{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}}{\eta_d(\theta_0)}\eta_{d-i}(A)E^*_0, \tag{25}$$

$$\eta^*_i(A^*)E_dE^*_0 = \frac{\varphi_d\varphi_{d-1}\cdots\varphi_{d-i+1}}{\tau_d(\theta_d)}\tau_{d-i}(A)E^*_0, \tag{26}$$

$$\tau^*_i(A^*)E_0E^*_d = \frac{\varphi_1\varphi_2\cdots\varphi_i}{\eta_d(\theta_0)}\eta_{d-i}(A)E^*_d, \tag{27}$$

$$\tau^*_i(A^*)E_dE^*_d = \frac{\phi_1\phi_2\cdots\phi_i}{\tau_d(\theta_d)}\tau_{d-i}(A)E^*_d. \tag{28}$$

Lemma 11.4 [22, Theorem 5.6]. *Let $\big(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d\big)$ denote a Leonard system and let $\big(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\big)$ denote the corresponding parameter array. Then*

$$E_0E^*_dE_dE^*_0 = \frac{\varphi_1\varphi_2\cdots\varphi_d}{\tau_d(\theta_d)\tau^*_d(\theta^*_d)}E_0E^*_0. \tag{29}$$

Lemma 11.5. *Let $\big(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d\big)$ denote a Leonard system with switching element S and dual switching element S^*. Then the following* (i)–(iv) *hold*:

(i) *SE^*_0 is equal to each of*

$$\frac{\tau_d(\theta_d)\tau^*_d(\theta^*_d)}{\varphi_1\varphi_2\cdots\varphi_d}E^*_dE_dE^*_0, \qquad \frac{\eta_d(\theta_0)\tau^*_d(\theta^*_d)}{\varphi_1\varphi_2\cdots\varphi_d}E^*_dE_0E^*_0. \tag{30}$$

(ii) *$S^{-1}E^*_d$ is equal to each of*

$$\frac{\tau_d(\theta_d)\eta^*_d(\theta^*_0)}{\phi_1\phi_2\cdots\phi_d}E^*_0E_dE^*_d, \qquad \frac{\eta_d(\theta_0)\eta^*_d(\theta^*_0)}{\phi_1\phi_2\cdots\phi_d}E^*_0E_0E^*_d. \tag{31}$$

(iii) *S^*E_0 is equal to each of*

$$\frac{\tau^*_d(\theta^*_d)\tau_d(\theta_d)}{\varphi_1\varphi_2\cdots\varphi_d}E_dE^*_dE_0, \qquad \frac{\eta^*_d(\theta^*_0)\tau_d(\theta_d)}{\varphi_1\varphi_2\cdots\varphi_d}E_dE^*_0E_0. \tag{32}$$

(iv) *$S^{*-1}E_d$ is equal to each of*

$$\frac{\tau^*_d(\theta^*_d)\eta_d(\theta_0)}{\phi_1\phi_2\cdots\phi_d}E_0E^*_dE_d, \qquad \frac{\eta^*_d(\theta^*_0)\eta_d(\theta_0)}{\phi_1\phi_2\cdots\phi_d}E_0E^*_0E_d. \tag{33}$$

Proof. We first show that SE^*_0 is equal to the expression on the left in (30). By Theorem 6.7, we have $SE^*_0V = E^*_dV$ so $SE^*_0 = E^*_dSE^*_0$. The element $E^*_dE_dE^*_0$ is nonzero by (29) and Lemma

4.7, so it forms a basis for $E^*_d\mathscr{A}E^*_0$. This space contains SE^*_0 so there exists $\alpha \in \mathbb{K}$ such that $SE^*_0 = \alpha E^*_d E_d E^*_0$. To find α, note that $E_0 S = E_0$ by (10) so $E_0 E^*_0 = \alpha E_0 E^*_d E_d E^*_0$. Comparing this with (29) we find

$$\alpha = \frac{\tau_d(\theta_d)\tau^*_d(\theta^*_d)}{\varphi_1\varphi_2\cdots\varphi_d}.$$

We have now shown that SE^*_0 is equal to the expression on the left in (30). To obtain the remaining assertions, apply D_4 and use Lemma 4.6 and Theorem 5.4. □

Lemma 11.6. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a Leonard system with switching element S and dual switching element S^*. Then for $0 \leqslant i \leqslant d$*

$$S^*S^{-1}S^{*-1}S\tau_i(A)E^*_0 = \frac{\phi_1\phi_2\cdots\phi_i}{\varphi_1\varphi_2\cdots\varphi_i}\frac{\varphi_1\varphi_2\cdots\varphi_{d-i}}{\phi_1\phi_2\cdots\phi_{d-i}}\tau_i(A)E^*_0. \tag{34}$$

Proof. We evaluate the expression on the left in (34). Recall S, A commute; pull S to the right past $\tau_i(A)$. Now evaluate SE^*_0 using the expression on the left in (30) and in the resulting expression evaluate $\tau_i(A)E^*_d E_d$ using (24); we find the left-hand side of (34) is a scalar multiple of

$$S^*S^{-1}S^{*-1}\tau^*_{d-i}(A^*)E_d E^*_0. \tag{35}$$

In line (35) pull S^{*-1} to the right past $\tau^*_{d-i}(A^*)$. Now evaluate $S^{*-1}E_d$ using the expression on the left in (33) and in the resulting expression evaluate $\tau^*_{d-i}(A^*)E_0E^*_d$ using (27); this shows (35) is a scalar multiple of

$$S^*S^{-1}\eta_i(A)E^*_d E_d E^*_0. \tag{36}$$

In line (36) pull S^{-1} to the right past $\eta_i(A)$. Now evaluate $S^{-1}E^*_d$ using the expression on the right in (31) and in the resulting expression evaluate $\eta_i(A)E^*_0E_0$ using (21); this shows (36) is a scalar multiple of

$$S^*\eta^*_{d-i}(A^*)E_0E^*_dE_dE^*_0. \tag{37}$$

In line (37) pull S^* to the right past $\eta^*_{d-i}(A^*)$. Now evaluate $S^*E_0E^*_dE_d$ using the expression on the left in (33) and in the resulting expression evaluate $\eta^*_{d-i}(A^*)E_dE^*_0$ using (26); this shows (37) is a scalar multiple of $\tau_i(A)E^*_0$. By the above comments we find that the left-hand side of (34) is a scalar multiple of $\tau_i(A)E^*_0$. Keeping track of the scalar we routinely verify (34). □

Theorem 11.7. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a Leonard system in $\mathscr{A}$ with parameter array $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$, switching element S and dual switching element S^*. Then the following* (i)–(iv) *hold for $0 \leqslant i \leqslant d$:*

(i) *The eigenvalue of $S^*S^{-1}S^{*-1}S$ on the ith component of $[0^*D]$ is*

$$\frac{\phi_1\phi_2\cdots\phi_i}{\varphi_1\varphi_2\cdots\varphi_i}\frac{\varphi_1\varphi_2\cdots\varphi_{d-i}}{\phi_1\phi_2\cdots\phi_{d-i}}. \tag{38}$$

(ii) *The eigenvalue of $S^*SS^{*-1}S^{-1}$ on the ith component of $[D^*D]$ is*

$$\frac{\varphi_d\varphi_{d-1}\cdots\varphi_{d-i+1}}{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}}\frac{\phi_d\phi_{d-1}\cdots\phi_{i+1}}{\varphi_d\varphi_{d-1}\cdots\varphi_{i+1}}. \tag{39}$$

(iii) *The eigenvalue of* $S^{*-1}S^{-1}S^*S$ *on the* i*th component of* $[0^*0]$ *is*

$$\frac{\varphi_1\varphi_2\cdots\varphi_i}{\phi_1\phi_2\cdots\phi_i}\frac{\phi_1\phi_2\cdots\phi_{d-i}}{\varphi_1\varphi_2\cdots\varphi_{d-i}}. \tag{40}$$

(iv) *The eigenvalue of* $S^{*-1}SS^*S^{-1}$ *on the* i*th component of* $[D^*0]$ *is*

$$\frac{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}}{\varphi_d\varphi_{d-1}\cdots\varphi_{d-i+1}}\frac{\varphi_d\varphi_{d-1}\cdots\varphi_{i+1}}{\phi_d\phi_{d-1}\cdots\phi_{i+1}}. \tag{41}$$

Proof. (i): Let ε_i denote the expression in (38). We show $S^*S^{-1}S^{*-1}S-\varepsilon_i I$ is zero on the ith component of $[0^*D]$. But this is immediate from Lemma 11.6 and since this ith component equals $\tau_i(A)E_0^*V$ by Lemma 8.8(i).

(ii)–(iv): Apply the D_4 action and use Lemma 4.6, Theorem 5.4. □

12. Representing the elements S, S^*, S^{-1}, S^{*-1} by matrices

Definition 12.1. Let $\left(A;\{E_i\}_{i=0}^d;A^*;\{E_i^*\}_{i=0}^d\right)$ denote a Leonard system in $\mathscr{A}$ and fix a nonzero $v_0^*\in E_0^*V$. By Lemma 8.8(i) the vectors $\tau_i(A)v_0^*$ $(0\leqslant i\leqslant d)$ form a basis for V. For $X\in\mathscr{A}$ let $X^\natural$ denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ that represents X with respect to this basis. We observe $\natural:\mathscr{A}\to\mathrm{Mat}_{d+1}(\mathbb{K})$ is an isomorphism of $\mathbb{K}$-algebras.

Example 12.2 [38, *Section* 21]. Let $\Phi=\left(A;\{E_i\}_{i=0}^d;A^*;\{E_i^*\}_{i=0}^d\right)$ denote a Leonard system in $\mathscr{A}$ and let the isomorphism $\natural:\mathscr{A}\to\mathrm{Mat}_{d+1}(\mathbb{K})$ be as in Definition 12.1. Then

$$A^\natural=\begin{pmatrix}\theta_0 & & & & & \mathbf{0}\\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d\end{pmatrix},\qquad A^{*\natural}=\begin{pmatrix}\theta_0^* & \varphi_1 & & & & \mathbf{0}\\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d\\ \mathbf{0} & & & & & \theta_d^*\end{pmatrix},$$

where $(\{\theta_i\}_{i=0}^d;\{\theta_i^*\}_{i=0}^d;\{\varphi_i\}_{i=1}^d;\{\phi_i\}_{i=1}^d)$ denotes the parameter array of Φ.

Let $\Phi=\left(A;\{E_i\}_{i=0}^d;A^*;\{E_i^*\}_{i=0}^d\right)$ denote a Leonard system in $\mathscr{A}$, with switching element S and dual switching element S^*. Our goal for this section is to find $S^\natural$, $(S^{-1})^\natural$, $S^{*\natural}$, $(S^{*-1})^\natural$.

Lemma 12.3. *Let* $\left(A;\{E_i\}_{i=0}^d;A^*;\{E_i^*\}_{i=0}^d\right)$ *denote a Leonard system with switching element* S *and dual switching element* S^*. *Then for* $0\leqslant i\leqslant d$

$$S\eta_{d-i}^*(A^*)E_d=\frac{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}}{\varphi_1\varphi_2\cdots\varphi_i}\tau_{d-i}^*(A^*)E_d, \tag{42}$$

$$S\eta_i^*(A^*)E_0=\frac{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}}{\varphi_1\varphi_2\cdots\varphi_i}\tau_i^*(A^*)E_0, \tag{43}$$

$$S^*\eta_{d-i}(A)E_d^*=\frac{\phi_1\phi_2\cdots\phi_i}{\varphi_1\varphi_2\cdots\varphi_i}\tau_{d-i}(A)E_d^*, \tag{44}$$

$$S^*\eta_i(A)E_0^* = \frac{\phi_1\phi_2\cdots\phi_i}{\varphi_1\varphi_2\cdots\varphi_i}\tau_i(A)E_0^*. \tag{45}$$

Proof. We first show (42). Using (23) we find that the left-hand side of (42) is a scalar multiple of

$$S\tau_i(A)E_0^*E_d. \tag{46}$$

In (46) pull S to the right past $\tau_i(A)$. Now evaluate SE_0^* using the expression on the left in (30) and in the resulting expression evaluate $\tau_i(A)E_d^*E_d$ using (24); this shows that (46) is a scalar multiple of

$$\tau^*_{d-i}(A^*)E_dE_0^*E_d. \tag{47}$$

By [38, Theorem 23.8]

$$E_0E_0^*E_0 = \frac{\phi_1\phi_2\cdots\phi_d}{\eta_d(\theta_0)\eta^*_d(\theta^*_0)}E_0.$$

Applying $\Downarrow$ to this and using Lemma 4.6 we find

$$E_dE_0^*E_d = \frac{\varphi_1\varphi_2\cdots\varphi_d}{\tau_d(\theta_d)\eta^*_d(\theta^*_0)}E_d.$$

Using this we find that (47) is a scalar multiple of $\tau^*_{d-i}(A^*)E_d$. By the above comments the left-hand side of (42) is a scalar multiple of $\tau^*_{d-i}(A^*)E_d$. Keeping track of the scalar we routinely verify (42). To obtain (43)–(45) apply D_4 to (42) using Lemma 4.6 and Theorem 5.4. □

Before we proceed we recall some scalars. Given a Leonard system $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ and given nonnegative integers r, s, t such that $r+s+t \leqslant d$, in [31, Definition 13.1] we defined a scalar $[r,s,t]_q \in \mathbb{K}$, where $q+q^{-1}+1$ is the common value of (9). For example, if $q \neq 1$ and $q \neq -1$ then

$$[r,s,t]_q = \frac{(q;q)_{r+s}(q;q)_{r+t}(q;q)_{s+t}}{(q;q)_r(q;q)_s(q;q)_t(q;q)_{r+s+t}},$$

where

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

We mention some features of $[r,s,t]_q$ that we will use. By [31, Lemma 13.2] we find $[r,s,t]_q$ is symmetric in r, s, t. We also have the following.

Lemma 12.4. *Referring to Definition 4.2, for $0 \leqslant j \leqslant d$ we have*

$$\tau_j = \sum_{i=0}^{j}[i, j-i, d-j]_q\tau_{j-i}(\theta_d)\eta_i, \tag{48}$$

$$\eta_j = \sum_{i=0}^{j}[i, j-i, d-j]_q\eta_{j-i}(\theta_0)\tau_i, \tag{49}$$

$$\tau^*_j = \sum_{i=0}^{j}[i, j-i, d-j]_q\tau^*_{j-i}(\theta^*_d)\eta^*_i, \tag{50}$$

$$\eta_j^* = \sum_{i=0}^{j} [i, j-i, d-j]_q \eta_{j-i}^*(\theta_0^*)\tau_i^*. \tag{51}$$

Proof. This is a routine consequence of [31, Theorem 15.2]. $\square$

Lemma 12.5. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system with switching element S. Then for $0 \leqslant j \leqslant d$*

$$S\tau_j(A) = \sum_{i=j}^{d} \frac{[j, i-j, d-i]_q \phi_d\phi_{d-1}\cdots\phi_{d-j+1}\tau_{i-j}^*(\theta_d^*)}{\varphi_1\varphi_2\cdots\varphi_i}\tau_i(A), \tag{52}$$

$$S^{-1}\tau_j(A) = \sum_{i=j}^{d} \frac{[j, i-j, d-i]_q \varphi_1\varphi_2\cdots\varphi_j\eta_{i-j}^*(\theta_0^*)}{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}}\tau_i(A). \tag{53}$$

Proof. Concerning (52), let L (resp. R) denote the expression on the left (resp. right). We show $L = R$. To do this we first show that $LE_0^* = RE_0^*$. We evaluate LE_0^* using (26) and in the resulting expression evaluate $S\eta_{d-j}^*(A^*)E_d$ using (42); this shows LE_0^* is a scalar multiple of

$$\tau_{d-j}^*(A^*)E_dE_0^*. \tag{54}$$

Now in (54) evaluate $\tau_{d-j}^*(A^*)$ using (50) and in the resulting expression evaluate $\eta_i^*(A^*)E_dE_0^*$ using (26); this shows that (54) is a scalar multiple of

$$\sum_{i=0}^{d-j} [i, d-j-i, j]_q \varphi_d\varphi_{d-1}\cdots\varphi_{d-i+1}\tau_{d-j-i}^*(\theta_d^*)\tau_{d-i}(A)E_0^*.$$

In this expression we replace i by $d-i$ and find it is equal to

$$\sum_{i=j}^{d} [j, i-j, d-i]_q \varphi_d\varphi_{d-1}\cdots\varphi_{i+1}\tau_{i-j}^*(\theta_d^*)\tau_i(A)E_0^*. \tag{55}$$

So far we have shown that LE_0^* is a scalar multiple of (55). Keeping track of the scalar we routinely find $LE_0^* = RE_0^*$. Now $L = R$ by Lemma 4.7. To obtain (53), apply $\downarrow$ to (52) and recall $S^\downarrow = S^{-1}$ from Theorem 5.4. $\square$

Theorem 12.6. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathscr{A}$ and let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$ denote the corresponding parameter array. Let S denote the switching element for Φ and let the isomorphism $\natural : \mathscr{A} \to \mathrm{Mat}_{d+1}(\mathbb{K})$ be from Definition 12.1. Then each of $S^\natural$, $(S^{-1})^\natural$ is lower triangular. Moreover, for $0 \leqslant j \leqslant i \leqslant d$ their (i, j) entries are given as follows*:

$$S_{i,j}^\natural = \frac{[j, i-j, d-i]_q \phi_d\phi_{d-1}\cdots\phi_{d-j+1}\tau_{i-j}^*(\theta_d^*)}{\varphi_1\varphi_2\cdots\varphi_i},$$

$$(S^{-1})_{i,j}^\natural = \frac{[j, i-j, d-i]_q \varphi_1\varphi_2\cdots\varphi_j\eta_{i-j}^*(\theta_0^*)}{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}}.$$

Proof. Follows from Lemma 12.5 and Definition 12.1. □

Lemma 12.7. *Let $\left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system with dual switching element S^*. Then for $0 \leqslant j \leqslant d$*

$$S^*\tau_j(A)E_0^* = \sum_{i=0}^{j} \frac{[i, j-i, d-j]_q \phi_1\phi_2\cdots\phi_i\tau_{j-i}(\theta_d)}{\varphi_1\varphi_2\cdots\varphi_i}\tau_i(A)E_0^*, \tag{56}$$

$$S^{*-1}\tau_j(A)E_0^* = \sum_{i=0}^{j} \frac{[i, j-i, d-j]_q \varphi_1\varphi_2\cdots\varphi_j\eta_{j-i}(\theta_0)}{\phi_1\phi_2\cdots\phi_j}\tau_i(A)E_0^*. \tag{57}$$

Proof. First we show (56). In the left-hand side of (56) we evaluate $\tau_j(A)$ using (48) to find

$$S^*\tau_j(A)E_0^* = \sum_{i=0}^{j}[i, j-i, d-j]_q\tau_{j-i}(\theta_d)S^*\eta_i(A)E_0^*.$$

In this equation, we evaluate $S^*\eta_i(A)E_0^*$ using (45) and get (56).

Next we show (57). By (45)

$$S^{*-1}\tau_j(A)E_0^* = \frac{\varphi_1\varphi_2\cdots\varphi_j}{\phi_1\phi_2\cdots\phi_j}\eta_j(A)E_0^*.$$

In this equation, we evaluate $\eta_j(A)$ using (49) and get (57). □

Theorem 12.8. *Let $\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ denote a Leonard system in $\mathscr{A}$ and let $\left(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\right)$ denote the corresponding parameter array. Let S^* denote the dual switching element for Φ and let the isomorphism $\natural : \mathscr{A} \to \mathrm{Mat}_{d+1}(\mathbb{K})$ be from Definition 12.1. Then each of $S^{*\natural}$, $(S^{*-1})^\natural$ is upper triangular. Moreover, for $0 \leqslant i \leqslant j \leqslant d$ their (i, j) entries are given as follows*:

$$S^{*\natural}_{i,j} = \frac{[i, j-i, d-j]_q\phi_1\phi_2\cdots\phi_i\tau_{j-i}(\theta_d)}{\varphi_1\varphi_2\cdots\varphi_i},$$

$$(S^{*-1})^\natural_{i,j} = \frac{[i, j-i, d-j]_q\varphi_1\varphi_2\cdots\varphi_j\eta_{j-i}(\theta_0)}{\phi_1\phi_2\cdots\phi_j}.$$

Proof. Follows from Lemma 12.7 and Definition 12.1. □

13. Leonard pairs in matrix form

In this section, we restate Theorems 12.6 and 12.8 in more concrete terms. Let us consider the following situation.

Definition 13.1. Let d denote a nonnegative integer. Let A and A^* denote matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$ of the form

$$A = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta_0^* & \varphi_1 & & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix}, \tag{58}$$

where

$$\theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \qquad (0 \leqslant i, j \leqslant d),$$
$$\varphi_i \neq 0 \quad (1 \leqslant i \leqslant d).$$

Observe A (resp. A^*) is multiplicity-free, with eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$). For $0 \leqslant i \leqslant d$ let E_i (resp. E_i^*) denote the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*).

Lemma 13.2. *Referring to Definition* 13.1, *the following* (i), (ii) *are equivalent.*

(i) *The pair A, A^* is a Leonard pair.*
(ii) *The sequence $\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d\right)$ is a Leonard system.*

Suppose (i), (ii) *hold. Then the sequence $\{\varphi_i\}_{i=1}^d$ is the first split sequence of Φ. Moreover, the isomorphism $\natural$ from Definition* 12.1 *is the identity map.*

Proof. The equivalence of (i), (ii) is established in [35, Lemma 6.2]. By [32, Theorem 14.3], the sequence $\{\varphi_i\}_{i=1}^d$ is the first split sequence of Φ. Comparing Example 12.2 with (58) we find $A^\natural = A$ and $A^{*\natural} = A^*$. Now $\natural$ is the identity map since A, A^* generate $\mathrm{Mat}_{d+1}(\mathbb{K})$ by [37, Corollary 5.5]. □

Theorem 13.3. *Referring to Lemma* 13.2, *assume the equivalent conditions* (i), (ii) *hold, and let S denote the switching element element for Φ. Then each of S, S^{-1} is lower triangular. Moreover, for $0 \leqslant j \leqslant i \leqslant d$ their (i, j) entries are given as follows*:

$$S_{i,j} = \frac{[j, i-j, d-i]_q \phi_d \phi_{d-1} \cdots \phi_{d-j+1} \tau_{i-j}^*(\theta_d^*)}{\varphi_1 \varphi_2 \cdots \varphi_i},$$
$$S_{i,j}^{-1} = \frac{[j, i-j, d-i]_q \varphi_1 \varphi_2 \cdots \varphi_j \eta_{i-j}^*(\theta_0^*)}{\phi_d \phi_{d-1} \cdots \phi_{d-i+1}}.$$

In the above lines $\{\phi_i\}_{i=1}^d$ is the second split sequence of Φ.

Proof. Combine Theorem 12.6 with Lemma 13.2. □

Theorem 13.4. *Referring to Lemma* 13.2, *assume the equivalent conditions* (i), (ii) *hold, and let S^* denote the dual switching element element for Φ. Then each of S^*, S^{*-1} is upper triangular. Moreover, for $0 \leqslant i \leqslant j \leqslant d$ their (i, j) entries are given as follows*:

$$S_{i,j}^* = \frac{[i, j-i, d-j]_q \phi_1 \phi_2 \cdots \phi_i \tau_{j-i}(\theta_d)}{\varphi_1 \varphi_2 \cdots \varphi_i},$$

$$S^{*-1}_{i,j} = \frac{[i, j-i, d-j]_q \varphi_1\varphi_2\cdots\varphi_j \eta_{j-i}(\theta_0)}{\phi_1\phi_2\cdots\phi_j}.$$

In the above lines $\{\phi_i\}_{i=1}^{d}$ *is the second split sequence of* Φ.

Proof. Combine Theorem 12.8 with Lemma 13.2. □

14. A characterization of a Leonard system in terms of the switching element

In this section, we give a characterization of a Leonard system in terms of its switching element. This characterization is a variation on [35, Theorem 6.3] and is stated as follows.

Theorem 14.1. *Referring to Definition* 13.1, *the following* (i), (ii) *are equivalent*:

(i) *The pair* A, A^* *is a Leonard pair.*
(ii) *There exists an invertible* $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ *and there exist nonzero scalars* $\phi_i \in \mathbb{K}$ $(1 \leqslant i \leqslant d)$ *such that* $X^{-1}AX = A$ *and*

$$X^{-1}A^*X = \begin{pmatrix} \theta^*_d & \phi_d & & & & \mathbf{0} \\ & \theta^*_{d-1} & \phi_{d-1} & & & \\ & & \theta^*_{d-2} & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \phi_1 \\ \mathbf{0} & & & & & \theta^*_0 \end{pmatrix}. \tag{59}$$

Suppose (i), (ii) *hold. Then* $\Phi = \left(A; \{E_i\}_{i=0}^{d}; A^*; \{E^*_i\}_{i=0}^{d}\right)$ *is a Leonard system,* $\{\phi_i\}_{i=1}^{d}$ *is the second split sequence of* Φ, *and* X *is a scalar multiple of the switching element for* Φ.

Proof. For notational convenience we abbreviate $V = \mathbb{K}^{d+1}$. For $0 \leqslant i \leqslant d$ let e_i denote the vector in V with ith coordinate 1 and all other coordinates 0. Observe that $\{e_i\}_{i=0}^{d}$ is a basis for V. From the form of A in (58) we find

$$(A - \theta_i I)e_i = e_{i+1} \quad (0 \leqslant i \leqslant d-1), \qquad (A - \theta_d I)e_d = 0. \tag{60}$$

From the form of A^* in (58) we find

$$(A^* - \theta^*_i I)e_i = \varphi_i e_{i-1} \quad (1 \leqslant i \leqslant d), \qquad (A^* - \theta^*_0 I)e_0 = 0. \tag{61}$$

(i) ⇒ (ii): By Lemma 13.2 the sequence $\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E^*_i\}_{i=0}^{d})$ is a Leonard system in $\mathrm{Mat}_{d+1}(\mathbb{K})$, with eigenvalue sequence $\{\theta_i\}_{i=0}^{d}$, dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^{d}$, and first split sequence $\{\varphi_i\}_{i=1}^{d}$. Let $\{\phi_i\}_{i=1}^{d}$ denote the second split sequence of Φ and let S denote the switching element for Φ. Then S^{-1} exists by Lemma 5.3. Also S commutes with A by Theorem 6.7 so $S^{-1}AS = A$. We now show that (59) holds with $X = S$. From (60) we find

$$e_i = \tau_i(A)e_0 \qquad (0 \leqslant i \leqslant d). \tag{62}$$

From the equation on the right in (61) we find $A^*e_0 = \theta^*_0 e_0$; using this and $e_0 \neq 0$ we find e_0 is a basis for E^*_0V. By this and (61) and (62) we have

$$(A^* - \theta^*_i I)\tau_i(A)E^*_0 = \varphi_i \tau_{i-1}(A)E^*_0 \quad (1 \leqslant i \leqslant d), \qquad (A^* - \theta^*_0 I)E^*_0 = 0. \tag{63}$$

We apply $\downarrow$ to (63) and use Lemma 4.6 to find

$$(A^* - \theta^*_{d-i} I)\tau_i(A)E^*_d = \phi_{d-i+1}\tau_{i-1}(A)E^*_d \quad (1 \leqslant i \leqslant d), \qquad (A^* - \theta^*_d I)E^*_d = 0. \tag{64}$$

By Theorem 6.7 and since $e_0 \in E^*_0 V$ we find $Se_0 \in E^*_d V$. Combining this with (64) we have

$$(A^* - \theta^*_{d-i} I)\tau_i(A)Se_0 = \phi_{d-i+1}\tau_{i-1}(A)Se_0 \quad (1 \leqslant i \leqslant d), \qquad (A^* - \theta^*_d I)Se_0 = 0. \tag{65}$$

Evaluating (65) using $S^{-1}AS = A$ and (62) we routinely find

$$(S^{-1}A^*S - \theta^*_{d-i} I)e_i = \phi_{d-i+1}e_{i-1} \quad (1 \leqslant i \leqslant d), \qquad (S^{-1}A^*S - \theta^*_d I)e_0 = 0.$$

By this we find (59) holds with $X = S$, as desired.

(ii) $\Rightarrow$ (i): We show $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ is a Leonard system in $\mathrm{Mat}_{d+1}(\mathbb{K})$. To do this we invoke [35, Theorem 5.1]. According to that theorem it suffices to display a decomposition $\{U_i\}_{i=0}^d$ of V such that

$$(A - \theta_i I)U_i = U_{i+1} \quad (0 \leqslant i \leqslant d-1), \qquad (A - \theta_d I)U_d = 0, \tag{66}$$

$$(A^* - \theta^*_i I)U_i = U_{i-1} \quad (1 \leqslant i \leqslant d), \qquad (A^* - \theta^*_0 I)U_0 = 0 \tag{67}$$

and a decomposition $\{V_i\}_{i=0}^d$ of V such that

$$(A - \theta_i I)V_i = V_{i+1} \quad (0 \leqslant i \leqslant d-1), \qquad (A - \theta_d I)V_d = 0, \tag{68}$$

$$(A^* - \theta^*_{d-i} I)V_i = V_{i-1} \quad (1 \leqslant i \leqslant d), \qquad (A^* - \theta^*_d I)V_0 = 0. \tag{69}$$

Define $U_i = \mathrm{Span}\{e_i\}$ for $0 \leqslant i \leqslant d$. The sequence $\{U_i\}_{i=0}^d$ is a decomposition of V since $\{e_i\}_{i=0}^d$ is a basis for V. The decomposition $\{U_i\}_{i=0}^d$ satisfies (66) by (60). The decomposition $\{U_i\}_{i=0}^d$ satisfies (67) by (61) and since $\varphi_i \neq 0$ for $1 \leqslant i \leqslant d$. Now define $V_i = \mathrm{Span}\{Xe_i\}$ for $0 \leqslant i \leqslant d$. The sequence $\{V_i\}_{i=0}^d$ is a decomposition of V since X^{-1} exists, and since $\{e_i\}_{i=0}^d$ is a basis for V. The decomposition $\{V_i\}_{i=0}^d$ satisfies (68) by (60) and since $X^{-1}AX = A$. The decomposition $\{V_i\}_{i=0}^d$ satisfies (69) by (59) and since $\phi_i \neq 0$ for $1 \leqslant i \leqslant d$. We have now verified (66)–(69) so [35, Theorem 5.1] applies; by that theorem Φ is a Leonard system in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Now the pair A, A^* is a Leonard pair by Lemma 13.2.

Now suppose (i), (ii) hold. We mentioned in the proof of (i) $\Rightarrow$ (ii) that $\Phi = \big(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d\big)$ is a Leonard system in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Next we show that X is a scalar multiple of the switching element S for Φ. To do this we invoke Theorem 6.7. Let $\mathscr{D}$ denote the subalgebra of $\mathrm{Mat}_{d+1}(\mathbb{K})$ generated by A. The element X commutes with A so $X \in \mathscr{D}$. By the left-most column in (59) we find $X^{-1}A^*Xe_0 = \theta^*_d e_0$ so $Xe_0 \in E^*_d V$. But e_0 is a basis for $E^*_0 V$ so $XE^*_0 V \subseteq E^*_d V$. Now X is a scalar multiple of S by Theorem 6.7. We saw in the proof of (i) $\Rightarrow$ (ii) that the sequence $\{\phi_i\}_{i=1}^d$ is the second split sequence of Φ. $\square$

The following is a variation in [36, Theorem 3.2].

Theorem 14.2. *Let d denote a nonnegative integer and let*

$$\Big(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\Big) \tag{70}$$

denote a sequence of scalars taken from $\mathbb{K}$. Assume this sequence satisfies the conditions (PA1) *and* (PA2) *in Theorem 4.5. Then the following* (i), (ii) *are equivalent*:

(i) *The sequence* (70) *satisfies* (PA3)–(PA5) *in Theorem* 4.5.
(ii) *There exists an invertible* $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ *such that*

$$X^{-1}\begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix} X = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix},$$

$$X^{-1}\begin{pmatrix} \theta^*_0 & \varphi_1 & & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & & \\ & & \theta^*_2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix} X = \begin{pmatrix} \theta^*_d & \phi_d & & & & \mathbf{0} \\ & \theta^*_{d-1} & \phi_{d-1} & & & \\ & & \theta^*_{d-2} & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \phi_1 \\ \mathbf{0} & & & & & \theta^*_0 \end{pmatrix}.$$

Proof. (i) $\Rightarrow$ (ii): By Theorem 4.5 there exists a Leonard system $\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d\right)$ over $\mathbb{K}$ that has parameter array (70). Let $\natural$ denote the corresponding isomorphism from Definition 12.1. Applying $\natural$ to each term in Φ if necessary, we may assume Φ is in $\mathrm{Mat}_{d+1}(\mathbb{K})$, and that $\natural$ is the identity map. Now A, A^* are of the form (58). Now (ii) holds by Theorem 14.1.

(ii) $\Rightarrow$ (i): Follows from Theorems 14.1 and 4.5. □

Note 14.3. We comment on how the switching element is related to the matrix G that appears in [35, Theorem 6.3] and [36, Theorem 3.2]. In [35, Theorem 6.3] reference is made to a Leonard pair A, A^* of the form (58); let $\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d\right)$ denote the corresponding Leonard system from Lemma 13.2. Then G is a nonzero scalar multiple of $S^{*-1}Y$, where S^* denotes the dual switching element for Φ and Y denotes the diagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ whose (i, i) entry is

$$\frac{\phi_1\phi_2\cdots\phi_i}{\varphi_1\varphi_2\cdots\varphi_i}$$

for $0 \leqslant i \leqslant d$.

Proof. From the data given in [35, Theorem 6.3] one readily verifies that YG^{-1} is invertible and commutes with A^*. Moreover, $YG^{-1}E_0V \subseteq E_dV$ where $V = \mathbb{K}^{d+1}$. Now applying Theorem 6.7 to Φ^* we find YG^{-1} is a nonzero scalar multiple of S^*. So G is a nonzero scalar multiple of $S^{*-1}Y$. □

15. Open problems

In this section, $\Phi = \left(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d\right)$ denotes a Leonard system in $\mathscr{A}$ with switching element S and dual switching element S^*. We will be discussing the flags

$$[z] \quad (z \in \Omega) \tag{71}$$

from Definition 8.2.

Definition 15.1. For distinct $x, y \in \Omega$, by a *switching element of type* (x, y) we mean an element X in $\mathscr{A}$ that sends $[x]$ to $[y]$ and fixes the remaining two flags in (71).

Example 15.2. By Theorem 9.1 we find that for nonzero $X \in \mathscr{A}$

(i) X is a switching element of type $(0^*, D^*)$ if and only if X is a scalar multiple of S.
(ii) X is a switching element of type $(D^*, 0^*)$ if and only if X is a scalar multiple of S^{-1}.
(iii) X is a switching element of type $(0, D)$ if and only if X is a scalar multiple of S^*.
(iv) X is a switching element of type $(D, 0)$ if and only if X is a scalar multiple of S^{*-1}.

Problem 15.3. Find a necessary and sufficient condition on the parameter array of Φ for there to exist a switching element of type $(0^*, D)$.

Problem 15.4. Find a necessary and sufficient condition on the parameter array of Φ for there to exist a switching element of type $(0^*, D)$ and a switching element of type $(0, D^*)$.

Note 15.5. For certain Φ there exists a second Leonard system $\Phi' = (B; \{F_i\}_{i=0}^d; B^*; \{F_i^*\}_{i=0}^d)$ in $\mathscr{A}$ such that the decomposition $\{F_i V\}_{i=0}^d$ coincides with $[0D^*]$ and the decomposition $\{F_i^* V\}_{i=0}^d$ coincides with $[0^* D]$. See for example [5]. In this case the switching element for Φ' is a switching element for Φ of type $(0^*, D)$, and the dual switching element for Φ' is a switching element for Φ of type $(0, D^*)$.

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